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Eigenvalues of the transversal Dirac operator on Kähler foliations

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Abstract

In this paper, we prove Kirchberg-type inequalities for any Kähler spin foliation. Their limiting-cases are then characterized as being transversal minimal Einstein foliations. The key point is to introduce the transversal Kählerian twistor operators.

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1. Introduction

On a compact Riemannian spin manifold (M^n, g_M), Th. Friedrich [1] showed that any eigenvalue λ of the Dirac operator satisfies

$$\lambda^2 \geq \frac{n}{4(n-1)} S_0, \tag{1.1}$$

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where S_0 denotes the infimum of the scalar curvature of M. The limiting case in (1.1) is characterized by the existence of a *Killing spinor*. As a consequence M is Einstein. Kirchberg [4] established that, on such manifolds any eigenvalue λ satisfies the inequalities

$$\lambda^2 \geq \begin{cases} \dfrac{m+1}{4m} S_0 & \text{if } m \text{ is odd,} \\ \dfrac{m}{4(m-1)} S_0 & \text{if } m \text{ is even.} \end{cases}$$

On a compact Riemannian spin foliation $(M, g_M, \mathcal{F})$ of codimension q with a bundle-like metric g_M such that the mean curvature κ is a basic coclosed 1-form, Jung [13] showed that any eigenvalue λ of the transversal Dirac operator satisfies

$$\lambda^2 \geq \frac{q}{4(q-1)} K_0^\nabla, \tag{1.2}$$

where $K_0^\nabla = \inf_M(\sigma^\nabla + |\kappa|^2)$, here σ^∇ denotes the transversal scalar curvature with the transversal Levi–Civita connection ∇. The limiting case in (1.2) is characterized by the fact that $\mathcal{F}$ is minimal ($\kappa = 0$) and transversally Einstein (see Theorem 3.1). The main result of this paper is the following.

Theorem 1.1. *Let $(M, g_M, \mathcal{F})$ be a compact Riemannian manifold with a Kähler spin foliation $\mathcal{F}$ of codimension $q = 2m$ and a bundle-like metric g_M. Assume that κ is a basic coclosed* 1*-form, then any eigenvalue λ of the transversal Dirac operator satisfies*:

$$\lambda^2 \geq \frac{m+1}{4m} K_0^\nabla \quad \text{if } m \text{ is odd} \tag{1.3}$$

and

$$\lambda^2 \geq \frac{m}{4(m-1)} K_0^\nabla \qquad \text{if } m \text{ is even.} \tag{1.4}$$

The limiting case in (1.3) is characterized by the fact that the foliation is minimal and by existence of a transversal Kählerian Killing spinor (see Theorem 4.3). We refer to Theorem 4.4 for the equality case in (1.4).

We point out that inequality (1.3) was proved by Jung [14] with the additional assumption that κ is *transversally holomorphic*. The author would like to thank Oussama Hijazi for his support.

2. Foliated manifolds

In this section, we summarize some standard facts about foliations. For more details, we refer to [8,13].

Let (M, g_M) be a $(p+q)$-dimensional Riemannian manifold and a foliation $\mathcal{F}$ of codimension q and let ∇^M be the Levi–Civita connection associated with g_M. We consider the exact sequence

$$0 \longrightarrow L \stackrel{\iota}{\longrightarrow} TM \stackrel{\pi}{\longrightarrow} Q \longrightarrow 0,$$

where L is the tangent bundle of TM and $Q = TM/L \simeq L^{\perp}$ the normal bundle. We assume g_M to be a *bundle-like metric* on Q, that means the induced metric g_Q verifies the holonomy invariance condition:

$$\mathcal{L}_X g_Q = 0 \quad \forall X \in \Gamma(L),$$

where $\mathcal{L}_X$ is the Lie derivative with respect to X. Let ∇ be the connection on Q defined by:

$$\nabla_X s = \begin{cases} \pi[X, Y_s] & \forall X \in \Gamma(L), \\ \pi(\nabla^M_X Y_s) & \forall X \in \Gamma(L^{\perp}), \end{cases}$$

where $s \in \Gamma(Q)$ and Y_s is the unique vector of $\Gamma(L^{\perp})$ such that $\pi(Y_s) = s$. The connection ∇ is metric and torsion-free. The curvature of ∇ acts on $\Gamma(Q)$ by:

$$R^{\nabla}(X, Y)s = \nabla_X \nabla_Y s - \nabla_Y \nabla_X s - \nabla_{[X,Y]} s \quad \forall X, Y \in \chi(M).$$

The transversal Ricci curvature is defined by:

$$\rho^{\nabla} : \Gamma(Q) \to \Gamma(Q), \qquad X \mapsto \rho^{\nabla}(X) = \sum_{j=1}^{q} R^{\nabla}(X, e_j)e_j.$$

Also, we define the transversal scalar curvature:

$$\sigma^{\nabla} = \sum_{i=1}^{q} g_Q(\rho^{\nabla}(e_i), e_i) = \sum_{i,j=1}^{q} R^{\nabla}(e_i, e_j, e_j, e_i),$$

where $\{e_i\}_{i=1,\ldots,q}$ is a local orthonormal frame of Q and $R^{\nabla}(X, Y, Z, W) = g_Q(R^{\nabla}(X, Y)Z, W)$, for all $X, Y, Z, W \in \Gamma(Q)$. The foliation $\mathcal{F}$ is said to be transversally Einstein if and only if

$$\rho^{\nabla} = \frac{1}{q}\sigma^{\nabla}\mathrm{Id}$$

with constant transversal scalar curvature. The mean curvature of Q is given by:

$$\kappa(X) = g_Q(\tau, X) \quad \forall X \in \Gamma(Q),$$

where $\tau = \sum_{l=1}^{p} II(e_l, e_l)$, with $\{e_l\}_{l=1,\dots,p}$ is a local orthonormal frame of $\Gamma(L)$ and II is the second fundamental form of $\mathcal{F}$ defined by:

$$II : \Gamma(L) \times \Gamma(L) \to \Gamma(Q), \qquad (X, Y) \mapsto II(X, Y) = \pi(\nabla^M_X Y).$$

We define basic r-forms by:

$$\Omega^r_B(\mathcal{F}) = \{\Phi \in \Lambda^r T^*M \,|\, X \lrcorner \Phi = 0 \text{ and } X \lrcorner \, \mathrm{d}\Phi = 0, \ \forall X \in \Gamma(L)\},$$

where d is the exterior derivative and $X\lrcorner$ is the interior product. Any $\Phi \in \Omega^r_B(\mathcal{F})$ can be locally written as

$$\sum_{1 \le j_1 < \cdots < j_r \le q} \beta_{j_1,\dots,j_r} \, \mathrm{d}y_{j_1} \wedge \cdots \wedge \mathrm{d}y_{j_r},$$

where $(\partial/\partial x_l)\beta_{j_1,\dots,j_r} = 0 \, \forall l = 1, \dots, p$. With the local expression of basic r-forms, one can verify that κ is closed if $\mathcal{F}$ is isoparametric ($\kappa \in \Omega^1_B(\mathcal{F})$). For all $r \ge 0$:

$$\mathrm{d}(\Omega^r_B(\mathcal{F})) \subset \Omega^{r+1}_B(\mathcal{F}).$$

We denote by $d_B = d|_{\Omega_B(\mathcal{F})}$ where $\Omega_B(\mathcal{F})$ is the tensor algebra of $\Omega^r_B(\mathcal{F})$. We have the following formulas:

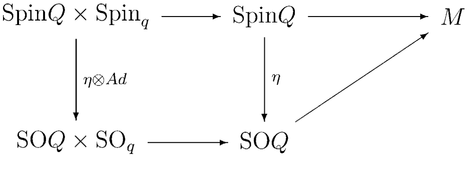
$$d_B = \sum_{i=1}^{q} e_i^\star \wedge \nabla_{e_i} \quad \text{and} \quad \delta_B = -\sum_{i=1}^{q} e_i \lrcorner \nabla_{e_i} + \kappa \lrcorner,$$

where δ_B is the adjoint operator of d_B with respect to the induced scalar product and $\{e_i\}_{i=1,\dots,q}$ is a local orthonormal frame of Q.

3. The transversal Dirac operator on Kähler Foliations

In this section, we start by recalling some facts on Riemannian foliations which could be found in [9–11,13]. For completeness, we also sketch a straightforward proof of inequality (1.2) established in [13] and end by recalling well-known facts (see [4,5,2,3,14]) on Kähler spin foliations.

On a foliated Riemannian manifold $(M, g_M, \mathcal{F})$, a transversal spin structure is a pair $(\mathrm{Spin}Q, \eta)$ where $\mathrm{Spin}Q$ is a Spin_q-principal fibre bundle over M and η a 2-fold cover such that the following diagram commutes:



The maps $\mathrm{Spin}Q \times \mathrm{Spin}_q \to \mathrm{Spin}Q$, and $\mathrm{SO}Q \times SO_q \longrightarrow \mathrm{SO}Q$, are, respectively, the actions of Spin_q and SO_q on the principal fibre bundles $\mathrm{Spin}Q$ and $\mathrm{SO}Q$. In this case, $\mathcal{F}$ is called a transversal spin foliation. We define the foliated spinor bundle by: $S(\mathcal{F}) := \mathrm{Spin}Q \times_\rho \Sigma_q$, where $\rho : \mathrm{Spin}_q \to \mathrm{Aut}\left(\Sigma_q\right)$, is the complex spin representation and Σ_q is a $\mathbb{C}$ vector space of dimension N with $N = 2^{[q/2]}$, where [ ] stands for the integer part. Recall that the Clifford multiplication $\mathcal{M}$ on $S(\mathcal{F})$ is given by:

$$\mathcal{M} : \Gamma(Q) \times \Gamma(S(\mathcal{F})) \to \Gamma(S(\mathcal{F})), \qquad (X, \Psi) \mapsto X \cdot \Psi.$$

There is a natural Hermitian product on $S(\mathcal{F})$ such that, for all $X, Y \in \Gamma(Q)$, the following relations are true:

$$\langle X \cdot \Psi, \Phi \rangle = -\langle \Psi, X \cdot \Phi \rangle, \qquad X(\langle \Psi, \Phi \rangle) = \langle \nabla_X \Psi, \Phi \rangle + \langle \Psi, \nabla_X \Phi \rangle,$$
$$\nabla_Y (X \cdot \Psi) = (\nabla_Y X) \cdot \Psi + X \cdot (\nabla_Y \Psi),$$

where ∇ is the Levi–Civita connection on $S(\mathcal{F})$ and $\Psi, \Phi \in \Gamma(S(\mathcal{F}))$.

The transversal Dirac operator [9,10] is locally given by:

$$D_{\mathrm{tr}}\Psi = \sum_{i=1}^{q} e_i \cdot \nabla_{e_i}\Psi - \frac{1}{2}\kappa \cdot \Psi \tag{3.1}$$

for all $\Psi \in \Gamma(S(\mathcal{F}))$. We can easily prove using Green's theorem [12] that this operator is formally self adjoint. Furthermore, in [10] it is proved that if $\mathcal{F}$ is isoparametric and $\delta_B\kappa = 0$, then we have the Schrödinger–Lichnerowicz formula:

$$D_{\mathrm{tr}}^2\Psi = \nabla_{\mathrm{tr}}^\star \nabla_{\mathrm{tr}}\Psi + \frac{1}{4}K_\sigma^\nabla \Psi,$$

where $K_\sigma^\nabla = \sigma^\nabla + |\kappa|^2$ and

$$\nabla_{\mathrm{tr}}^\star \nabla_{\mathrm{tr}}\Psi = -\sum_{i=1}^{q} \nabla_{e_i,e_i}^2 \Psi + \nabla_\kappa \Psi$$

with $\nabla_{X,Y}^2 = \nabla_X \nabla_Y - \nabla_{\nabla_X Y}$, for all $X, Y \in \Gamma(TM)$.

Denote by $\mathcal{P}$ the transversal twistor operator defined by

$$\mathcal{P} : \Gamma(S(\mathcal{F})) \xrightarrow{\nabla^{\mathrm{tr}}} \Gamma(Q^* \otimes S(\mathcal{F})) \xrightarrow{\pi} \Gamma(\ker \mathcal{M}),$$

where π is the orthogonal projection on the kernel of the Clifford multiplication $\mathcal{M}$. With respect to a local orthonormal frame $\{e_1, \ldots, e_q\}$, for all $\Psi \in \Gamma(S(\mathcal{F}))$, one has

$$\mathcal{P}\Psi = \sum_{i=1}^{q} e_i^* \otimes \left(\nabla_{e_i}\Psi + \frac{1}{q} e_i \cdot D_{\mathrm{tr}}\Psi + \frac{1}{2q} e_i \cdot \kappa \cdot \Psi \right). \tag{3.2}$$

For any spinor field Ψ, one can easily show that

$$\sum_{i=1}^{q} e_i \cdot \mathcal{P}_{e_i}\Psi = 0. \tag{3.3}$$

Now we give a simple proof of the following theorem.

Theorem 3.1 (Jung [13]). *Let $(M, g_M, \mathcal{F})$ be a compact Riemannian manifold with a spin foliation $\mathcal{F}$ of codimension q and a bundle-like metric g_M with $\kappa \in \Omega^1_B(\mathcal{F})$. Assume that $\delta_B\kappa = 0$ and let λ be an eigenvalue of the transversal Dirac operator, then*

$$\lambda^2 \geq \frac{q}{4(q-1)} K_0^{\nabla}. \tag{3.4}$$

Proof. For all $\Psi \in \Gamma(S(\mathcal{F}))$, we have using identities (3.2), (3.3) and (3.1):

$$|\mathcal{P}\Psi|^2 = |\nabla^{\mathrm{tr}}\Psi|^2 - \frac{1}{q}|D_{\mathrm{tr}}\Psi|^2 - \frac{1}{q}\Re(D_{\mathrm{tr}}\Psi, \kappa \cdot \Psi) - \frac{1}{4q}|\kappa|^2|\Psi|^2.$$

For any spinor field Φ, we have that $(\Phi, \kappa \cdot \Phi) = -(\kappa \cdot \Phi, \Phi) = -\overline{(\Phi, \kappa \cdot \Phi)}$, so the scalar product $(\Phi, \kappa \cdot \Phi)$ is a pure imaginary function. Hence for any eigenspinor Ψ of the transversal Dirac operator, we obtain

$$\int_M |\mathcal{P}\Psi|^2 + \frac{1}{4q}\int_M |\kappa|^2|\Psi|^2 = \int_M |\nabla^{\mathrm{tr}}\Psi|^2 - \frac{1}{q}\int_M \lambda^2|\Psi|^2$$

from which we deduce (3.4) with the help of the Schrödinger–Lichnerowicz formula. Finally, we can easily prove in the limiting case that $\mathcal{F}$ is minimal i.e. $\kappa = 0$, and transversally Einstein. □

A foliation $\mathcal{F}$ is called Kähler if there exists a complex parallel orthogonal structure $J : \Gamma(Q) \longrightarrow \Gamma(Q)$ ($\dim Q = q = 2m$). Let Ω be the associated Kähler, i.e., for all $X, Y \in \Gamma(Q)$, $\Omega(X, Y) = g_Q(J(X), Y) = -g_Q(X, J(Y))$. The Kähler form can be locally expressed as

$$\Omega = \frac{1}{2}\sum_{i=1}^{q} e_i \cdot J(e_i) = -\frac{1}{2}\sum_{i=1}^{q} J(e_i) \cdot e_i$$

and for all $X \in \Gamma(Q)$, we have $[\Omega, X] := \Omega \cdot X - X \cdot \Omega = 2J(X)$. Under the action of the Kähler form, the spinor bundle splits into an orthogonal sum

$$S(\mathcal{F}) = \bigoplus_{r=o}^{m} S_r(\mathcal{F}),$$

where $S_r(\mathcal{F})$ is an eigenbundle associated with the eigenvalue $i\mu_r = i(2r - m)$ of the Kähler form Ω. Moreover, the spinor bundle of a Kähler spin foliation carries a parallel anti-linear map j satisfying the relations:

$$j^2 = (-1)^{m(m+1)/2} Id, \qquad [X, j] = 0, \qquad (j\Psi, j\Phi) = (\Phi, \Psi)$$

and we have $j\Psi_r = (j\Psi)_{m-r}$. For all $X \in \Gamma(Q)$, we have

$$p_+(X) \cdot S_r(\mathcal{F}) \subset S_{r+1}(\mathcal{F}) \quad \text{and} \quad p_-(X) \cdot S_r(\mathcal{F}) \subset S_{r-1}(\mathcal{F}),$$

where $p_\pm(X) = (X \mp iJ(X))/2$. We define the operator $\tilde{D}_{\mathrm{tr}}$ by

$$\tilde{D}_{\mathrm{tr}}\Psi = \sum_{i=1}^{q} J(e_i) \cdot \nabla_{e_i}\Psi - \frac{1}{2} J(\kappa) \cdot \Psi.$$

The local expression of $\tilde{D}_{\mathrm{tr}}$ is independent of the choice of the local frame and by Green's theorem [12], we prove that this operator is self-adjoint. On a Kähler spin foliation, the operators D_{tr} and $\tilde{D}_{\mathrm{tr}}$ satisfy:

$$[\Omega, D_{\mathrm{tr}}] = 2\tilde{D}_{\mathrm{tr}}, \tag{3.5}$$

$$[\Omega, \tilde{D}_{\mathrm{tr}}] = -2D_{\mathrm{tr}}, \tag{3.6}$$

$$[\Omega, D_{\mathrm{tr}}^2] = 0, \tag{3.7}$$

$$D_{\mathrm{tr}}\tilde{D}_{\mathrm{tr}} + \tilde{D}_{\mathrm{tr}}D_{\mathrm{tr}} = 0, \tag{3.8}$$

$$\tilde{D}_{\mathrm{tr}}^2 = D_{\mathrm{tr}}^2. \tag{3.9}$$

We should point out that Eqs. (3.7)–(3.9) are true under the assumptions that $\mathcal{F}$ is isoparametric and $\delta_B\kappa = 0$. Now we define the two operators D_+ and D_- by

$$D_+ = \tfrac{1}{2}(D_{\mathrm{tr}} - i\tilde{D}_{\mathrm{tr}}) \quad \text{and} \quad D_- = \tfrac{1}{2}(D_{\mathrm{tr}} + i\tilde{D}_{\mathrm{tr}}). \tag{3.10}$$

Furthermore, D_{tr} splits into D_+ and D_-, and we have the two exact sequences:

$$\Gamma(S_m(\mathcal{F})) \xrightarrow{D_-} \cdots \Gamma(S_r(\mathcal{F})) \xrightarrow{D_-} \Gamma(S_{r-1}(\mathcal{F})) \xrightarrow{D_-} \cdots \Gamma(S_0(\mathcal{F})), \tag{3.11}$$

$$\Gamma(S_0(\mathcal{F})) \xrightarrow{D_+} \cdots \Gamma(S_r(\mathcal{F})) \xrightarrow{D_+} \Gamma(S_{r+1}(\mathcal{F})) \xrightarrow{D_+} \cdots \Gamma(S_m(\mathcal{F})). \tag{3.12}$$

4. Eigenvalues of the transversal Dirac operator

In this section, we prove Kirchberg-type inequalities by using the transversal Kählerian twistor operators on Kähler spin foliations. We refer to [7,6].

Definition 4.1. On a Kähler spin foliation, we define the transversal Kählerian twistor operators by

$$\mathcal{P}^{(r)} : \Gamma(S_r(\mathcal{F})) \xrightarrow{\nabla^{\mathrm{tr}}} \Gamma(Q^* \otimes S_r(\mathcal{F})) \xrightarrow{\pi_r} \Gamma(\ker \mathcal{M}_r),$$

where $\mathcal{M}_r$ is the transversal Clifford multiplication defined by

$$\mathcal{M}_r : \Gamma(Q^* \otimes S_r(\mathcal{F})) \to \Gamma(S_{r-1}(\mathcal{F})) \oplus \Gamma(S_{r+1}(\mathcal{F})),$$
$$X \otimes \Psi_r \mapsto p_-(X) \cdot \Psi_r \oplus p_+(X) \cdot \Psi_r.$$

For all $r \in \{0, \dots, m\}$ and $\Psi_r \in \Gamma(S_r(\mathcal{F}))$, we have

$$\mathcal{P}^{(r)}\Psi_r = \sum_{i=1}^{q} e_i^* \otimes (\nabla_{e_i}\Psi_r + a_r p_-(e_i) \cdot \mathcal{D}_+\Psi_r + b_r p_+(e_i) \cdot \mathcal{D}_-\Psi_r), \tag{4.1}$$

where $\mathcal{D}_\pm = D_\pm + (1/2)p_\pm(\kappa)$ with $a_r = 1/(2(r+1))$ and $b_r = 1/(2(m-r+1))$. For any spinor field $\Psi_r \in \Gamma(S_r(\mathcal{F}))$, we can easily prove

$$\sum_{i=1}^{q} e_i \cdot \mathcal{P}^{(r)}_{e_i}\Psi_r = 0. \tag{4.2}$$

Remark 4.2. For any non-zero eigenvalue λ of D_{tr}, there exists a spinor field $\Psi \in \Gamma(S(\mathcal{F}))$ called of type $(r, r+1)$, such that $D_{\mathrm{tr}}\Psi = \lambda\Psi$ and $\Psi = \Psi_r + \Psi_{r+1}$, with $r \in \{0, \dots, m-1\}$. By using (3.10)–(3.12) it follows that $D_-\Psi_r = D_+\Psi_{r+1} = 0$, $D_-\Psi_{r+1} = \lambda\Psi_r$, $D_+\Psi_r = \lambda\Psi_{r+1}$ and $\|\Psi_r\|_{L^2} = \|\Psi_{r+1}\|_{L^2}$.

Proof. Let φ be an eigenspinor of D_{tr}. There exists an r such that φ_r does not vanish. Let $\Psi = \frac{1}{\lambda}D_-D_+\varphi_r + D_+\varphi_r$, one can easily get that $D_{\mathrm{tr}}\Psi = \lambda\Psi$. □

Theorem 4.3. *Let $(M, g_M, \mathcal{F})$ be a compact Riemannian manifold with a Kähler spin foliation $\mathcal{F}$ of codimension $q = 2m$ and a bundle-like metric g_M with $\kappa \in \Omega^1_B(\mathcal{F})$ and $\delta_B\kappa = 0$. Then any eigenvalue λ of the transversal Dirac operator, satisfies*

$$\lambda^2 \geq \frac{m+1}{4m}K_0^\nabla. \tag{4.3}$$

If Ψ is an eigenspinor of type $(r, r+1)$ associated with an eigenvalue λ satisfying equality in (4.3), then $r = (m-1)/2$, the foliation $\mathcal{F}$ is minimal and for all $X \in \Gamma(Q)$, the spinor Ψ satisfies

$$\nabla_X\Psi + \frac{\lambda}{2(m+1)}(X \cdot \Psi - i\varepsilon J(X) \cdot \bar{\Psi}) = 0, \tag{4.4}$$

where $\varepsilon = (-1)^{(m-1)/2}$, and $\bar{\Psi} := (-1)^r(\Psi_r - \Psi_{r+1})$. As a consequence m is odd and $\mathcal{F}$ is transversally Einstein with non-negative constant transversal curvature σ^∇.

Proof. For all $\Psi_r \in \Gamma(S_r(\mathcal{F}))$, using identities (4.1) and (4.2), we have

$$|\mathcal{P}^{(r)}\Psi_r|^2 = \sum_{i=1}^{q} |\mathcal{P}^{(r)}_{e_i}\Psi_r|^2 = \sum_{i=1}^{q}(\mathcal{P}^{(r)}_{e_i}\Psi_r, \nabla_{e_i}\Psi_r)$$
$$= \sum_{i=1}^{q}(\nabla_{e_i}\Psi_r + a_r p_-(e_i) \cdot \mathcal{D}_+\Psi_r + b_r p_+(e_i) \cdot \mathcal{D}_-\Psi_r, \nabla_{e_i}\Psi_r).$$

Finally we obtain:

$$|\mathcal{P}^{(r)}\Psi_r|^2 = |\nabla^{\mathrm{tr}}\Psi_r|^2 - a_r|\mathcal{D}_+\Psi_r|^2 - b_r|\mathcal{D}_-\Psi_r|^2. \tag{4.5}$$

Let λ be an eigenvalue of D_{tr} and let Ψ an eigenspinor of type $(r, r+1)$. Applying equality (4.5) to Ψ_r, one gets

$$|\mathcal{P}^{(r)}\Psi_r|^2 = |\nabla^{\mathrm{tr}}\Psi_r|^2 - a_r\lambda^2|\Psi_{r+1}|^2 - a_r\lambda\Re(\Psi_{r+1}, p_+(\kappa)\cdot\Psi_r) - \frac{a_r}{4}|p_+(\kappa)\cdot\Psi_r|^2 - \frac{b_r}{4}|p_-(\kappa)\cdot\Psi_r|^2.$$

By the Schrödinger–Lichnerowicz formula and by the fact that Ψ_r and Ψ_{r+1} have the same L^2-norms, we get

$$\int_M |\mathcal{P}^{(r)}\Psi_r|^2 + \frac{a_r}{4}\int_M |p_+(\kappa)\cdot\Psi_r|^2 + \frac{b_r}{4}\int_M |p_-(\kappa)\cdot\Psi_r|^2 = \int_M \left((1-a_r)\lambda^2 - \frac{1}{4}K_\sigma^\nabla\right)|\Psi_r|^2 - a_r\lambda\int_M \Re(\Psi_{r+1}, p_+(\kappa)\cdot\Psi_r). \tag{4.6}$$

Similarly applying (4.5) to Ψ_{r+1}, we obtain

$$\int_M |\mathcal{P}^{(r+1)}\Psi_{r+1}|^2 + \frac{a_{r+1}}{4}\int_M |p_+(\kappa)\cdot\Psi_{r+1}|^2 + \frac{b_{r+1}}{4}\int_M |p_-(\kappa)\cdot\Psi_{r+1}|^2 = \int_M \left((1-b_{r+1})\lambda^2 - \frac{1}{4}K_\sigma^\nabla\right)|\Psi_{r+1}|^2 + b_{r+1}\lambda\int_M \Re(\Psi_{r+1}, p_+(\kappa)\cdot\Psi_r), \tag{4.7}$$

where $K_\sigma^\nabla = \sigma^\nabla + |\kappa|^2$. In order to get rid the term $\lambda\int_M \Re(\Psi_{r+1}, p_+(\kappa)\cdot\Psi_r)$, since the l.h.s. of (4.6) and (4.7) are non-negative, dividing (4.6) by a_r and (4.7) by b_{r+1} then summing up, we find by substituting the values of a_r and b_{r+1}:

$$\lambda^2 \geq \frac{m+1}{4m}K_0^\nabla.$$

Now, we discuss the limiting case of inequality (4.3). Dividing (4.6) by a_r and (4.7) by b_{r+1} then summing up as before, and substituting a_r, b_{r+1} and λ^2 by their values, we easily deduce that $\kappa = 0$, $\mathcal{P}^{(r)}\Psi_r = 0$ and $\mathcal{P}^{(r+1)}\Psi_{r+1} = 0$. Hence by (4.6), we find that $\lambda^2 = (1/4(1-a_r))\sigma_0 = (m+1/4m)\sigma_0$ where $\sigma_0 = \inf_M \sigma^\nabla$, then $r = (m-1)/2$ and m is odd. It remains to prove that Ψ satisfies (4.4). For $r = (m-1)/2$, by definition of the Kählerian twistor operators, for all $j \in \{1, \ldots, q\}$, we obtain

$$\nabla_{e_j}\Psi_r + \frac{\lambda}{m+1}p_-(e_j)\cdot\Psi_{r+1} = 0$$

and

$$\nabla_{e_j}\Psi_{r+1} + \frac{\lambda}{m+1}p_+(e_j)\cdot\Psi_r = 0.$$

Summing up the two equations, we get (4.4) for $X = e_j$. Using Ricci identity in (4.4), one easily proves that $\mathcal{F}$ is transversally Einstein. □

Theorem 4.4. *Under the same conditions as in Theorem 4.3 for m even, any eigenvalue* λ *of the transversal Dirac operator satisfies*

$$\lambda^2 \geq \frac{m}{4(m-1)} K_0^{\nabla}. \tag{4.8}$$

If Ψ *is an eigenspinor of type* $(r, r+1)$ *associated with an eigenvalue satisfying equality in* (4.8), *then* $r = m/2$, *the foliation* $\mathcal{F}$ *is minimal and* Ψ *satisfies for all* $X \in \Gamma(Q)$:

$$\nabla_X \Psi_{r+1} = -\frac{\lambda}{q}(X - iJX) \cdot \Psi_r. \tag{4.9}$$

Proof. Let Ψ an eigenspinor of type $(r, r+1)$ associated with any eigenvalue λ of the transversal Dirac operator D_{tr}. Recalling equalities (4.6) and (4.7), we have

$$0 \leq \int_M \left((1-a_r)\lambda^2 - \frac{1}{4}K_\sigma^{\nabla}\right)|\Psi_r|^2 - a_r\lambda \int_M \Re(\Psi_{r+1}, p_+(\kappa)\cdot\Psi_r) \tag{4.10}$$

and

$$0 \leq \int_M \left((1-b_{r+1})\lambda^2 - \frac{1}{4}K_\sigma^{\nabla}\right)|\Psi_{r+1}|^2 + b_{r+1}\lambda \int_M \Re(\Psi_{r+1}, p_+(\kappa)\cdot\Psi_r). \tag{4.11}$$

Hence if $\lambda \int_M \Re(\Psi_{r+1}, p_+(\kappa)\cdot\Psi_r) \leq 0$, then by (4.11)

$$\lambda^2 \geq \frac{1}{4(1-b_{r+1})} K_0^{\nabla},$$

The antilinear isomorphism j sends $S_r(\mathcal{F})$ to $S_{m-r}(\mathcal{F})$. This allows the choice of μ_r to be non-negative (i.e. $r \geq m/2$) where μ_r is the eigenvalue associated with Ψ_r. Then a careful study of the graph of the function $1/(1-b_{r+1})$, yields (4.8).

On the other hand if $\lambda \int_M \Re(\Psi_{r+1}, p_+(\kappa)\cdot\Psi_r) > 0$. Applying Eq. (4.5) to the spinor $j\Psi$, which is a spinor of type $(m-(r+1), m-r)$, we find the same inequalities as (4.10) and (4.11), then

$$\lambda^2 > \frac{1}{1-a_r}\frac{K_0^{\nabla}}{4}.$$

As before we can choose $\mu_{m-(r+1)} \geq 0$ (i.e. $r \leq \frac{m}{2} - 1$). A careful study of the graph of the function $1/(1-a_r)$ gives inequality (4.8).

Now we discuss the limiting case of (4.8). As we have seen, it could not be achieved if $\lambda \int_M \Re(\Psi_{r+1}, p_+(\kappa)\cdot\Psi_r) > 0$, so only the other case should be considered. By (4.7), one has

$$\int_M |\mathcal{P}^{(r+1)}\Psi_{r+1}|^2 + \frac{a_{r+1}}{4}\int_M |p_+(\kappa)\cdot\Psi_{r+1}|^2 + \frac{b_{r+1}}{4}\int_M |p_-(\kappa)\cdot\Psi_{r+1}|^2 - b_{r+1}\lambda \times \int_M \Re(\Psi_{r+1}, p_+(\kappa)\cdot\Psi_r) = (1-b_{r+1}) \times \int_M \left(\frac{m}{4(m-1)}K_0^{\nabla} - \frac{1}{4(1-b_{r+1})}K_\sigma^{\nabla}\right)|\Psi_{r+1}|^2. \tag{4.12}$$

Since $m/(m-1) = \inf_{r\geq m/2}(1/(1-b_{r+1}))$, and the l.h.s. of (4.12) is non-negative, we deduce that $\kappa = 0$, $\mathcal{P}^{r+1}\Psi_{r+1} = 0$ and $m/(m-1) = 1/(1-b_{r+1})$ so $r = m/2$. It remains to show that Eq. (4.9) holds. For this, take $X = e_j$ where $\{e_j\}_{j=1,\ldots,q}$ is a local orthonormal frame. For $r = m/2$, and by definition of the Kählerian twistor operators, for all $j \in \{1,\ldots,q\}$, we obtain

$$\nabla_{e_j}\Psi_{r+1} + \frac{\lambda}{q}(e_j - iJe_j)\cdot\Psi_r = 0. \quad \square$$

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